

# Scattering on Magnetic Monopoles

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The time-dependent scattering theory of charged particles on magnetic monopoles is investigated within a mathematical framework, which duly pays attention to the fact that the wave-functions of the scattered particles are sections in a non-trivial complex line-bundle. It is found that Möller operators have to be defined in a way which takes into account the peculiar long-range behaviour of the monopole field. Formulas for the scattering matrix and the differential cross-section are derived, and, as a by-product, a momentum space picture for particles, which are described by sections in the underlying complex line-bundle, is presented.

## Introduction

It has been recognized within the framework of geometric quantization [1, 2] that it is very convenient to interpret the wave-function of a particle as a section in a complex line-bundle. For the quantum-mechanical description of an electron in the field of a magnetic monopole, this interpretation is in fact essential [3, 4, 5] because it yields in the most natural way the mathematical explanation of the quantization of magnetic charge, first discovered by Dirac [6]. However, as we shall see, this new interpretation forbids to apply the standard methods of scattering theory without significant changes. It is the purpose of this note to demonstrate how these methods have to be changed and how a consistent theory of electron scattering on magnetic monopoles is finally obtained.

## 1. Standard Scattering Theory

In order to see what changes we have to perform, it is useful to recall first the two basic approaches of scattering theory [7]. To be as close as possible to the physical situation which we finally want to describe, let us first consider an electron moving in a time-independent magnetic field which is represented by a two-form  $B$ . Assume that

$$B = dA, \quad \left( A = \sum_{j=1}^3 A_j dx^j \right), \quad (1)$$

and consider the Schrödinger equation

$$E\psi = -\frac{1}{2m} \sum_{j=1}^3 \left( \frac{\partial}{\partial x^j} + iqA_j \right)^2 \psi, \quad (2)$$

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which is valid for a particle with mass  $m$  and charge  $q$ .

a) The prescription of the time-independent scattering theory reads then as follows: set

$$E = k^2/2m$$

and look for a solution  $\psi$ , of Eq. (2) which asymptotically ( $|x| \rightarrow \infty$ ) behaves as

$$\psi \cong e^{ikx} + A(\Omega) e^{i|k||x|/|x|}, \quad (3)$$

where  $A$  is function of the angle variables  $\Omega$  alone. The differential cross-section  $\sigma$  is then given by

$$\sigma = |A|^2.$$

b) In contrast to this approach, the time-dependent scattering theory [7] compares the time-evolution governed by the Hamiltonian

$$H = -\frac{1}{2m} \sum_{j=1}^3 \left( \frac{\partial}{\partial x^j} + iqA_j \right)^2 \quad (4)$$

with the evolution governed by the free Hamiltonian

$$H_0 = -\frac{1}{2m} \Delta. \quad (5)$$

To this end one considers the Möller-operators  $\Omega_{\pm}$ , which are given by the strong limits

$$\Omega_{\pm} = \lim_{t \rightarrow \pm\infty} e^{iHt} e^{-iH_0 t}, \quad (6)$$

and studies the so-called scattering matrix  $S$ , formally defined by

$$S = \Omega_+^* \Omega_-. \quad (7)$$

The physical importance of these operators lies in the following facts:

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Choose  $\psi \in L^2(R^3)$  arbitrarily and consider the time-evolution of  $\Omega_- \psi$ , i.e.

$$\psi(t) = e^{-iHt} \Omega_- \psi.$$

Then, by construction of  $\Omega_\pm$ , it follows that

$$\lim_{t \rightarrow -\infty} \|\psi(t) - e^{-iH_0 t} \psi\| = 0$$

which shows that  $\psi(t)$  behaves as  $e^{-iH_0 t} \psi$  when  $t \rightarrow -\infty$ . Hence, at large negative times, the time-evolution of  $\Omega_- \psi$  equals the free evolution of  $\psi$ . On the other hand we find that

$$\lim_{t \rightarrow +\infty} \|e^{-iH_0 t} S\psi - \psi(t)\| = 0 \quad (8)$$

which shows that, at large positive times,  $\psi(t)$  behaves as  $e^{-iH_0 t} S\psi$  i.e. we observe the free evolution of  $S\psi$ . If there is not magnetic field, the the equation

$$\psi(t) = e^{-iH_0 t} \psi$$

will hold for all times (and not just, when  $t \rightarrow -\infty$ ). Hence we write

$$\psi(t) = e^{-iH_0 t} \psi + \psi_{sc}(t) \quad (9)$$

in the presence of a nonvanishing field and interpret  $\psi_{sc}(t)$  as the scattering wave due to the interaction with the external field. Consequently, the probability that the particle is scattered into a cone  $C$  with apex at the origin, is given by the formula

$$P(C, \psi) = \lim_{t \rightarrow +\infty} \int_C d^3x |\psi_{sc}(t)(x)|^2. \quad (10)$$

Using Eq. (8) and some mathematical properties of the free evolution operator one finds [7]:

$$P(C, \psi) = \int_C d^3p |(S - 1) \hat{\psi}(p)|^2, \quad (11)$$

where  $\hat{\psi}$  denotes the Fourier transform of  $\psi$ . Equation (11) is known as the “scattering into cones formula”; it is shown in standard text-books, how the differential cross-section can be extracted out of it.

## 2. Electron-Monopole Scattering

Let us now consider electron scattering in the field of a magnetic monopole with magnetic charge  $\mu$  fixed at the origin; i.e. the electron moves in a magnetic field described by the two-form

$$\begin{aligned} B &= \mu B_0, \\ B_0 &= (x^1 dx^2 \wedge dx^3 + x^3 dx^1 \wedge dx^2 \\ &\quad + x^2 dx^3 \wedge dx^1) / |x|^3. \end{aligned} \quad (12)$$

$B$  is closed, but not exact; hence there is no vector-potential  $A$  in the domain  $R^3 - 0 =: \dot{R}^3$ , where  $B$  is well-defined. Therefore, Eq. (1) and, as a consequence, Eq. (2) seem to be mathematically meaningless. The way out of this difficulty was shown by Dirac. His arguments become, however, more transparent, when we use the language of modern differential geometry. In this language the quantum-mechanical description of a charged particle in a magnetic field  $B$  reads as follows [4]: The particle is described by a section in complex line-bundle  $\xi \xrightarrow{\pi} D(B)$ ; ( $D(B) = \dot{R}^3$  denotes the domain, where  $B$  is well-defined). The line bundle  $\xi$  must have the following properties:

a) There is a fibre metric  $\langle, \rangle$  and a covariant derivative  $\nabla$  in  $\xi$  which are compatible; i.e. for arbitrary sections  $\sigma_1, \sigma_2$  and vector fields  $X$  the equation

$$\begin{aligned} X(\langle \sigma_1, \sigma_2 \rangle) &= \langle \nabla_X \sigma_1, \sigma_2 \rangle \\ &\quad + \langle \sigma_1, \nabla_X \sigma_2 \rangle \end{aligned} \quad (13)$$

holds.

b) The curvature  $\omega(\nabla)$  and the external magnetic field  $B$  are related by the formula

$$\omega(\nabla) = iqB. \quad (14)$$

It has been shown in Ref. [4] that this new formulation fulfills all the requirements of quantum theory. In particular, if the particle is described by a section  $\sigma$ , then  $\langle \sigma, \sigma \rangle(x)$  represents the probability to find the particle at the point  $x$ . It follows that our quantum mechanical Hilbert space  $H$  consists of all square-integrable sections  $\sigma$ :

$$H = \left\{ \sigma; \int_{D(B)} d^3x \langle \sigma, \sigma \rangle(x) < \infty \right\}, \quad (15)$$

with scalar product

$$\langle \sigma_1, \sigma_2 \rangle = \int_{D(B)} d^3x \langle \sigma_1, \sigma_2 \rangle(x).$$

Moreover, the Schrödinger equation (2) reads now as follows:

$$E\sigma = -\frac{1}{2m} \sum_{j=1}^3 \nabla_j^2 \sigma. \quad (16)$$

(In this equation  $\nabla_j$  denotes the covariant derivative in the direction of the vector field  $\partial/\partial x_j$ .)

The two requirements a) and b) determine  $\xi$  (together with  $\langle, \rangle$  and  $\nabla$ ) up to strong bundle isomorphisms, provided  $D(B)$  is simply connected. Physically, such isomorphisms correspond to gauge

transformations and do not lead to observable consequences. More important is, therefore, the question, under which conditions on  $B$  a suitable line bundle  $\xi$  exists. The answer is extremely simple:  $\xi$  exists if and only if the cohomology class  $[qB/2\pi]$  defined by  $B$  is integer [2]. If, in particular,  $B$  is given by formula (12) the last statement requires

$$2q\mu = n \in \mathbb{Z}, \quad (17)$$

which is precisely Dirac's quantization of the monopole charge  $\mu$ .

Hence we will assume that (17) holds indeed with  $n \neq 0$  so that the general requirements of quantum theory are fulfilled. We must expect, of course, that the fact, that from now on we have to deal with sections in a nontrivial bundle, will present us new kinds of difficulties not known in ordinary Schrödinger theory. In particular, let us investigate what happens to the two standard approaches of scattering theory. Clearly, the time-independent approach a) cannot be used, because there is no way to formula a boundary condition like Eq. (3) for sections in a nontrivial line-bundle. This has been realized by several authors who have changed this condition [8, 9, 10]. We will come to this point later and consider first the time-dependent approach b). Also this approach is, in our case, no longer useful. The reason lies in the fact that we have to compare the time evolution governed by

$$H = -\frac{1}{2m} \sum_{j=1}^3 \nabla_j^2 \quad (18)$$

(which acts on sections in  $\xi$ ), with the time evolution given by

$$H_0 = -\frac{1}{2m} \Delta$$

(which acts on ordinary functions). Hence the Möller-operators can no longer be defined as in Eq. (6), because  $H$  and  $H_0$  act on completely different spaces.

At this point it is useful to recall that the definition (6) of the Möller operators is known to be no longer valid, when long range forces are present. In particular, Eq. (6) does not hold for the Coulomb problem [7]. In this case  $H_0$  has to be replaced by a "free" Hamiltonian  $H_0'$  which differs from the expression (5). Note, however, that in the Coulomb problem, the particular choice of  $H_0'$  is not predicted by the general theory, but has to be guessed

from the particular form of the corresponding Schrödinger-equation and its solutions [11]. This is now the procedure which we want to adopt for our problem, too; i.e. we want to find an operator  $H_0'$  replacing  $H_0$  in Eq. (6) which yields the Möller operators. We hope that  $H_0'$  will be somehow naturally determined by  $H$  itself, and consequently, the next sections deal with a more detailed investigation of  $H$ .

However, we have to mention before, that there is another subtle difficulty concealed in our problem, which arises from the "scattering into cones formula" (Eq. (11)). This formula contains the Fourier transform, or, in physical terms, the momentum distribution of the wave-function  $\psi$ . Obviously, sections in a nontrivial line bundle cannot be Fourier transformed like ordinary functions. Hence we expect for our particular scattering problem a "scattering into cones formula" where the Fourier transformation is replaced by a suitable unitary operator  $F$ . If  $\sigma$  is a section, we may, by obvious analogy, interpret  $F\sigma$  as the "momentum space wave-function" corresponding to  $\sigma$ . We shall see in the following, how  $F$  is determined by the appropriate choice of  $H_0'$ .

### 3. The Hamiltonian of Electron-Monopole Scattering

According to the last section, we have to construct a line bundle  $\xi_n \xrightarrow{\pi} \mathbb{R}^3$  with a fibre metric  $\langle, \rangle$  and a compatible covariant derivative  $\nabla$ , such that

$$\omega(\nabla) = -i(n/2)B_0, \quad (\text{compare Eq. (12)}),$$

with

$$2q\mu = n \in \mathbb{Z}.$$

This construction has been shown elsewhere [4]; at this place we just give the result without proofs.

Identify  $U(1)$  with the group of unimodular complex numbers and let  $d_n: U(1) \rightarrow GL(\mathbb{C})$  denote the representation  $d_n(z) = z^n$ ; ( $z \in U(1)$ ).

It turns out [4] that there is a fixed principal  $U(1)$ -bundle  $P \xrightarrow{\tilde{\pi}} \mathbb{R}^3$  such that  $\xi_n$  is associated to  $P$  via the representation  $d_n$  for all  $n$ ; i.e.

$$\xi_n = Px_n C. \quad (19)$$

Moreover, there is a connection form  $\alpha$  in  $P$ , such that

$$d\alpha = -\tilde{\pi}^* B_0/2. \quad (20)$$

Let  $\text{Sec}(n)$  and  $H(n)$  denote the vector spaces of all sections in  $\xi_n$  and the Hilbert space of all square-integrable sections, respectively (compare Eq. (15)); let  $F(n)$  denote the space of  $n$ -equivariant functions on  $P$ , i.e.

$$F(n) = \{\psi \mid P \rightarrow \mathbb{C}; \psi(q \cdot z) = z^{-n} \psi(q), \\ \text{for all } q \in P, z \in U(1)\}. \quad (21)$$

Let  $X$  be any vector field on  $\hat{R}^3$  and let  $H(X)$  its horizontal lift to  $P$  (induced by the connection form  $\alpha$ ).

By a standard theorem in differential geometry [12], there is a canonical linear isomorphism

$$d_n^\#: F(n) \rightarrow \text{Sec}(n);$$

moreover, a covariant derivative  $\nabla$  in  $\xi_n$  is given by the formula ( $\sigma \in \text{Sec}(n)$ ):

$$\nabla_X \sigma = d_n^\# H(X) (d_n^{\#-1} \sigma). \quad (22)$$

$\nabla$  is compatible with the fibre metric

$$\langle \sigma_1, \sigma_2 \rangle(x) =: (d_n^{\#-1} \sigma_1)(q) (d_n^{\#-1} \sigma_2)(q), \quad (23)$$

where  $q \in \tilde{\pi}^{-1}(x)$   $P$  is any point in the fibre over  $x \in \hat{R}^3$ . It is now convenient to define a suitable volume form  $\omega_4$  on  $P$ :

$$\omega_4 = \frac{1}{2\pi} \tilde{\pi}^* (\omega_3) \wedge \alpha, \quad (\omega_3 = d^3x), \quad (24)$$

and to introduce the Hilbert space  $\tilde{H}(n) \subset F(n)$ :

$$\tilde{H}(n) = \{\psi \in F(n); \int_P \bar{\psi} \psi \omega_4 < \infty\}, \quad (25)$$

with scalar product given by the formula

$$\langle \psi_1, \psi_2 \rangle = \int_P \bar{\psi}_1 \psi_2 \omega_4. \quad (26)$$

Then  $d_n^\#$  restricts to a Hilbert space isomorphism

$$d_n^\#: \tilde{H}(n) \rightarrow H(n).$$

According to Eq. (18), the Hamiltonian of electron-monopole scattering is given by

$$H = -\frac{1}{2m} \sum_{j=1}^3 \nabla_j^2.$$

In view of the definition of  $\nabla$  (compare Eq. (22)) it is more convenient to study not  $H$ , but

$$\tilde{H} = d_n^{\#-1} H d_n^\# = -\frac{1}{2m} \sum_{j=1}^3 H(e_j)^2 \quad (27)$$

which acts on  $\tilde{H}(n)$  ( $e_j$  denotes a unit vector in the direction of the  $j$ -th coordinate axis). Since  $d_n^\#$  is a Hilbert space isomorphism we loose nothing in

doing so, but have the advantage of dealing with an ordinary partial differential operator.

It remains to specify  $P \xrightarrow{\tilde{\pi}} \hat{R}^3$  together with  $\alpha$ . Let  $Q$  denote the algebra of quaternions, with orthonormal basis  $e_0, e_1, e_2, e_3$ . ( $e_0$  is the unit element, and the elements  $e_1, e_2, e_3$  generate the usual multiplication table.) Set

$$P = \dot{Q} = Q - 0 \quad (28)$$

and identify  $\hat{R}^3$  with the elements  $q \in \dot{Q}$  orthogonal to  $e_0$ . If  $z \in U(1)$  has the form  $z = \alpha + i\beta$ , define, for all  $q \in \dot{Q}$

$$q \cdot z = q(\alpha + e_3 \beta). \quad (29)$$

Let  $\tilde{\pi}: P \rightarrow \hat{R}^3$  denote the map given by

$$\tilde{\pi}(q) = q e_3 \bar{q}; \quad (30)$$

by (28), (29) and (30) we have then indeed defined the  $U(1)$ -principal bundle  $P$ . The connection form  $\alpha$  is given by

$$\alpha_q(h_q) = \langle q e_3, h_q \rangle / |q|^2 \quad (31)$$

(for arbitrary tangent vectors  $h_q$  at  $q$ ).

By a straightforward calculation, one finds

$$\omega_4 = \frac{4}{\pi} |q|^2 d^4q \quad (32)$$

and

$$\tilde{H} = -\frac{1}{8m|q|^2} (\Delta_4 - n^2/4 |q|^2). \quad (33)$$

In the derivation of (33), it has been used that  $\tilde{H}$  acts on  $n$ -equivariant function; therefore,  $n$  appears in this formula. ( $\Delta_4$  denotes the Laplacian in four dimensions.)

According to Sect. 2, we have to look for an operator  $H_0'$  which replaces  $H_0$  Equation (6).  $H_0'$  has to fulfill the following physical requirements:

- a) There should be no charge-dependence in  $H_0$ .
- b) At large distances from the origin  $H$  and  $H_0$  should look more or less equal.

Now observe that  $|q|^2 = |\tilde{\pi}(q)|$ . Hence, at large distances from the origin, we find from Equation (33):

$$\tilde{H} \rightarrow \tilde{H}_0 = -\frac{1}{8m|q|^2} \Delta_4. \quad (34)$$

Moreover,  $\tilde{H}_0$  shows no explicit dependence on  $n$ , that is on the charge  $q$ . There is, of course, some

hidden  $n$ -dependence, in  $\tilde{H}_0$ , because it is assumed that it acts on  $n$ -equivariant functions; but note that, for all  $n$ ,  $\tilde{H}_0$  has the same form. The fact that  $\tilde{H}_0$  acts on  $n$ -equivariant functions has, however, a certain effect on the corresponding angular momentum spectrum (compare Ref. [4], [8]): the allowed angular momenta have the quantized values  $j = |n/2|, |n/2| + 1, \dots$  etc., i.e. they are modified by the field of the monopole. This phenomenon is already well-known from the classical theory [13], [14]; moreover, in the classical theory, one shows that this modification persists even at large, (and in fact infinite) distances from the origin. This physical effect will, therefore, be taken into account if we make the following choice for  $H_0'$ :

$$H_0' = d_n^\# \tilde{H}_0 d_n^{\#-1}. \quad (35)$$

Besides the physical arguments leading to (35), it should, however, be stressed, that (35) represents from a formal point of view, the only choice for  $H_0'$  which appears to be mathematically natural.

#### 4. The Scattering into Cones Formula

After having defined  $H$  and  $H_0'$ , we are now able to study the Möller operators  $\Omega_\pm$  given by Equation (6). From the last section it is clearly more convenient to work within the Hilbert space  $\tilde{H}(n)$ , of  $n$ -equivariant functions on  $P$ , and to investigate the operators

$$\tilde{\Omega}_\pm = d_n^{\#-1} \Omega_\pm d_n^\# = \lim_{t \rightarrow \pm \infty} e^{iHt} e^{-i\tilde{H}_0 t}. \quad (36)$$

To this end we need some technical preparations. First recall that the space of unit quaternions forms the group  $SU(2)$ . The Wigner coefficients [15]  $D_{m,m'}^j(q_0)$ , ( $q_0 \in SU(2)$ ), of the standard irreducible representations of  $SU(2)$  in  $C^{2j+1}$ , yield (by the Peter-Weil theorem) [16], a complete basis for the Hilbert space of square-integrable functions on  $S^3$  (with standard volume form). Consider the function  $\psi: P \rightarrow \mathbb{C}$

$$\psi(q) = \varphi(r) D_{m,m'}^j(q|q|), \quad (37)$$

where  $r = |q|^2$  and  $\varphi$  has compact support on the positive real line.  $\psi$  is  $n$ -equivariant, if and only if

$$m' = -n/2 \quad (38)$$

(this follows from the well-known properties of the Wigner coefficients [15]). (38) implies [15] that

$$j = |m'|, |m'| + 1, |m'| + 1, \dots,$$

and that  $m$  has the values  $m = -j, -j+1, \dots, j$ . Assume now that (38) holds. It follows from the completeness of the Wigner coefficients, that finite sums of functions which have the form (37) generate a dense subspace  $\hat{H}(n)$  of  $\tilde{H}(n)$ . Let now  $p \in \mathbb{R}$  be greater than  $-1/2$  and define

$$h_p(\varphi)(r) = \int_0^\infty dr' J_p(rr') \sqrt{rr'} \varphi(r'). \quad (39)$$

$J_p$  denotes a Bessel function and  $h_p$  is known as the Hankel transformation [17] of order  $p$ . Define  $T$  and  $T_0$  by the formulas

$$\begin{aligned} (T\psi)(q) &= (t_j \varphi)(r) \cdot D_{m,m'}^j(q|q|), \\ (T_0\psi)(q) &= (t_j^0 \varphi)(r) \cdot D_{m,m'}^j(q|q|), \end{aligned} \quad (40)$$

where  $t_j$  and  $t_j^0$  are given by the equations

$$\begin{aligned} t_j \varphi &= \frac{1}{r} h_{\lambda(j)}(r \varphi), \\ t_j^0 \varphi &= \frac{1}{r} h_{\lambda_0(j)}(r \varphi), \end{aligned}$$

with

$$\begin{aligned} \lambda(j) &= ((j + \tfrac{1}{2})^2 - m'^2)^{1/2}, \\ \lambda_0(j) &= j + \tfrac{1}{2}. \end{aligned}$$

It is well-known [17] that  $h_p$  extends to a unitary transformation of the Hilbert space of all square-integrable functions on the positive real line. Using this fact one can easily prove that the operators  $T$  and  $T_0$  extend to unitary transformations of  $\tilde{H}(n)$ .

Moreover, one can show that  $T$  and  $T_0$  are hermitian and idempotent, i.e.

$$T^2 = T_0^2 = \text{id}.$$

By partial integration, one proves furthermore that the equations

$$\begin{aligned} (T\tilde{H}\psi)(q) &= \frac{1}{2m} r^2 (T\psi)(q), \\ (T_0\tilde{H}_0\psi)(q) &= \frac{1}{2m} r^2 (T_0\psi)(q), \end{aligned} \quad (42)$$

hold for any function of the form (37) and hence for any  $\psi \in \hat{H}(n)$ . Since  $\hat{H}(n)$  is dense in  $\tilde{H}(n)$ , Eq. (42) shows that the differential operators  $H$  and  $\tilde{H}_0$  have self-adjoint extensions which are unitarily equivalent to the operator of multiplication with  $r^2/2m$ . Hence both operators have a continuous spectrum, (the positive real line), and no bound states.



In addition to the operators  $T$  and  $T_0$  we define for functions  $\psi$  of the form (37), (compare Eq. (40)),

$$\begin{aligned} U_{\pm} \psi &= \exp \{ \pm i \pi (\lambda(j) - \lambda_0(j))/2 \} \cdot \psi, \\ U_0 \psi &= \exp \{ -i (\lambda_0(j) + 1)/2 \} \cdot \psi. \end{aligned} \quad (43)$$

The operators  $U_{\pm}$  and  $U_0$  extend trivially to isometries of  $\hat{H}(n)$ , and hence of  $\tilde{H}(n)$ .

After these preparations we can present the result for the limits (36):

$$\tilde{Q}_{\pm} = U_{\pm} T T_0. \quad (44)$$

(44) yields the scattering matrix  $\tilde{S}$  (defined in the space of equivariant functions),

$$\tilde{S} = \tilde{Q}_{+}^{*} \tilde{Q}_{-} = U_{-}^2. \quad (45)$$

With the help of  $d_n^{\#}$  we obtain the  $S$ -matrix in the space  $H(n)$ , of sections in  $\xi_n$ :

$$S = d_n^{\#} \tilde{S} d_n^{\#-1}. \quad (46)$$

With the help of (36) and (46) one can then establish the "scattering into cones formula" which holds for our problem. The derivation is completely analogous to the one of Sect. 2; the only difference consists in the replacement of the ordinary wave-function  $\psi$  by a section  $\sigma$  in the bundle  $\xi_n$ . One finds that the probability of a particle being scattered into the cone  $C$  is given by the formula:

$$\begin{aligned} P(C, \sigma) &= \int_C d^3x \langle (S - 1) F \sigma, \\ &\quad (S - 1) F \sigma \rangle(x) \end{aligned} \quad (47)$$

with

$$F \sigma = d_n^{\#} U_0 T_0 d_n^{\#-1}. \quad (48)$$

The operator  $F$  replaces the ordinary Fourier transformation, as was anticipated in Section 2.

The differential cross section of electron-monopole scattering can be derived from (47). For completeness we state the result here. Let  $\theta$  denote the scattering angle and  $P$  the momentum of the charged particle; the differential cross section  $d\sigma/d\Omega$  is given by the equation

$$d\sigma/d\Omega = |2f(\theta)/P| \quad (49)$$

with

$$\begin{aligned} f(\theta) &= \left[ (\sin \theta)^{-1/2} \frac{d^2}{d\theta^2} (\sin \theta)^{1/2} \right. \\ &\quad \left. + \frac{1}{16(\sin \theta/2)^2} + \frac{1 - 4n^2}{16(\cos \theta/2)^2} \right] f_0(\theta) \end{aligned}$$

and

$$\begin{aligned} f_0(\theta) &= \sum_{k=0}^{\infty} \frac{T(k)}{2k + |n| + 1} \\ &\quad \cdot P_k^{(0, |n|)}(\cos \theta) (\cos \theta/2)^{|n|}. \end{aligned}$$

$P_k^{(0, |n|)}$  denotes a Jacobi polynomial and  $T(k)$  is given by

$$\begin{aligned} T(k) &= -1 + \exp \{ -i \pi (|n|/2 + k + 1/2)^2 \\ &\quad - (|n|/2)^2 ]^{1/2} - k - |n|/2 - 1/2 \} \}. \end{aligned} \quad (50)$$

The proofs of the Eqs. (44)–(49) are given in the appendix.

## Discussion

In the last section we have shown, that after some reasonable changes in the definition of the Möller operators, the time-dependent approach to scattering theory can be successfully applied to the electron-monopole scattering problem, despite of the fact, that wave-functions have to be replaced by sections in a complex line bundle. Actually all the physically relevant quantities can even be computed in closed form.

We have mentioned in Sect. 2 that our problem was already treated within a modified time-independent approach by several authors [8, 9, 10]. Their result can be more easily compared to ours if we consider phase shifts, which can be read off from Equation (45). As a function of angular momentum  $j$ ,  $j = |n|/2, |n|/(2+1), \dots$  etc., the phase shifts  $\delta(j)$  are of the form (compare also Equation (50):

$$\begin{aligned} \delta(j) &= -\pi [(j + \tfrac{1}{2})^2 - (|n|/2)^2]^{1/2} \\ &\quad - j - \tfrac{1}{2}. \end{aligned} \quad (51)$$

In contrast to this, one finds in the references mentioned before

$$\delta(j) = -\pi [(j + \tfrac{1}{2})^2 - (|n|/2)^2]^{-1/2}. \quad (52)$$

The expression (50) has the (physically reasonable) property that  $\delta(j)$  vanishes for large angular momenta  $j$ , whereas (51) diverges. As a consequence the corresponding expression for the differential cross section is a divergent series which has to be regularized ad hoc. Our expression for the cross section behaves perfectly well in this respect.

We have already mentioned that the time-independent approach uses a modified boundary condition at infinity. It might be that this new boundary

condition is mathematically not adequate, but that the numerical regularization (which is in fact done on the computer, see Ref. [9] compensates for this defect. Our result could then be compared only numerically with the time-independent approach presented so far in the literature; no effort in this direction has yet been done.

## Appendix

We want to prove now the Eqs. (44)–(49) of Sect. 4 (the notation is the same as in the preceding text). For this purpose we define the unitary operators  $U_1$ ,  $U_2$  acting on elements  $\psi \in \tilde{H}(n)$  by the formula, ( $q \in P$ ):

$$(U_1(\alpha)\psi)(q) = \exp(i\alpha|q|)^4 \psi(q); \quad \alpha \in R; \quad (\text{A1})$$

$$(U_2(\alpha)\psi)(q) = \alpha^{3/2} \psi(\alpha^{1/2}q); \quad \alpha \in R, \quad \alpha > 0.$$

In addition, we define the unitary operators (compare Section 4):

$$\tilde{\varphi}(r) = \lim_{\varepsilon \rightarrow 0^+} r^{-1/2} \int_0^\infty dr' \left\{ J_{\lambda_0(j)}(rr') \cdot \exp\{-r'^2(\varepsilon \pm it/2m)\} \int_0^\infty dr'' J_{\lambda_0(j)}(r'r'') r''^{3/2} \varphi(r'') \right\}.$$

(The limit  $\varepsilon \rightarrow 0^+$  has to be performed in order to extend the Hankeltransformation to functions which do not have compact support [17].) Since  $\varphi$  has compact support, the order of integrations can be exchanged; the result of the integral over  $\sigma'$  is then known explicitly [18]:

$$\tilde{\varphi}(r) = r^{-1/2} \cdot \frac{m}{t} \exp\{\pm i\pi/2(\lambda_0(j) + 1)\} \int_0^\infty dr'' r''^{3/2} \exp\left\{\pm i \frac{m}{2t}(r^2 + r''^2)\right\} J_{\lambda_0(j)}(rr''m/t) \varphi(r'').$$

(13) follows now immediately from this equation. The proof of (14) proceeds by the same argument, with  $\tilde{H}$  replacing  $\tilde{H}_0$  and  $\lambda(j)$  replacing  $\lambda_0(j)$ .

*Lemma 2.*

$$\lim_{t \rightarrow \infty} \exp\{\mp i\tilde{H}_0 t\} - V_\pm^0(t) = 0, \quad (\text{A5})$$

$$\lim_{t \rightarrow \infty} \exp\{\pm i\tilde{H} t\} - V_\pm(t) = 0. \quad (\text{A6})$$

*Proof.* It follows from Lemma 1 that for all  $\psi \in \tilde{H}(n)$

$$\begin{aligned} & |\exp\{\mp i\tilde{H}_0 t\} \psi - V_\pm^0(t) \psi| \\ &= |\exp\{\mp i\tilde{H} t\} \psi - V_\pm(t) \psi| \\ &= |U_1(\mp m/2t) \psi - \psi|. \end{aligned}$$

Now the right hand side converges to 0 as  $t \rightarrow \infty$ , since  $U(\alpha)$  converges strongly to the identity as  $\alpha \rightarrow 0$ . Now set

$$\tilde{\Omega}_\pm = U_\pm T T_0.$$

$$V_\pm^0(t) = U_2(m/t) U_1(\mp t/2m) U_0^{\pm 1} T_0, \quad (\text{A2})$$

$$V_\pm(t) = U_2(m/t) U_1(\mp t/2m) U_\mp U_0^{\pm 1} T,$$

with  $t \in R$  and  $t > 0$ .

We proof now

*lemma 1.* For all  $t > 0$  the following equations hold:

$$\exp\{\mp i\tilde{H}_0 t\} = V_\pm^0(t) U_1(\mp m/2t), \quad (\text{A3})$$

$$\exp\{\mp i\tilde{H} t\} = V_\pm(t) U_1(\mp m/2t). \quad (\text{A4})$$

*Proof.*  $U_1$ ,  $U_2$  and hence  $V_\pm^0$ ,  $V_\pm$  are all unitary operators on  $H(m)$ . Therefore, we need to prove (A3) and (A4) only for functions of the form

$$\psi(q) = \varphi(r) D_{m,m'}^j(q/|q|)$$

with  $\varphi$  having compact support on the positive real line, because such functions generate a dense subspace of  $\tilde{H}(n)$ .

Equation (42) yields for such a function:

$$(\exp\{\mp i\tilde{H}_0 t\} \psi)(q) = \tilde{\varphi}(r) D_{mm'}^j(q/|q|),$$

where

We proof

$$\text{lemma 3. } \tilde{\Omega}_\pm = \lim_{t \rightarrow \pm \infty} \exp\{\pm i\tilde{H} t\} \exp\{\mp i\tilde{H}_0 t\},$$

*proof.* For any  $\psi \in \tilde{H}(n)$  we have the inequalities

$$\begin{aligned} & |\tilde{\Omega}_\pm \psi - \exp\{\pm i\tilde{H} t\} \exp\{\mp i\tilde{H}_0 t\} \psi| \\ &= |\exp\{\mp i\tilde{H} t \tilde{\Omega}_\pm\} \psi - \exp\{\mp i\tilde{H} t\} \psi| \\ &\leq |\exp\{\mp i\tilde{H} t \tilde{\Omega}_\pm\} \psi - V_\pm(t) \tilde{\Omega}_\pm \psi| \\ &\quad + |\exp\{\mp i\tilde{H}_0 t\} \psi - V_\pm^0(t) \psi| \\ &\quad + |V_\pm(t) \tilde{\Omega}_\pm \psi - V_\pm^0(t) \psi|. \end{aligned} \quad (\text{A7})$$

According to Lemma 2, the first two terms converge to zero as  $t \rightarrow \infty$  and the definitions of  $V_\pm(t)$  and  $V_\pm(t)$  imply that the last term satisfies

$$\begin{aligned} & |V_\pm(t) \tilde{\Omega}_\pm \psi - V_\pm^0(t) \psi| \\ &= |U_\mp U_0^{\pm 1} T U_\pm T T_0 \psi - U_0^{\pm 1} T_0 \psi|, \end{aligned}$$

which vanishes since the operators  $T$ ,  $U_{\pm}$ ,  $U_0^{\pm 1}$  all commute and

$$U_{\mp} U_{\pm} = T^2 = T_0^2 = 1,$$

(compare Equation (41)). Hence we have proven the lemma. Lemma 3 shows that the Möller operators for our scattering problem are given by the formula

$$\Omega_{\pm} = d_n^{\#} \tilde{\Omega}_{\pm} d_n^{\#-1} \quad (\text{A8})$$

(compare Section 3). Using again  $T^2 = T_0^2 = 1$  together with Lemma 3 we find that the scattering matrix  $S$  is explicitly given by the formula

$$S = \Omega_{+}^{\dagger} \Omega_{-} = d_n^{\#} U_{-}^2 d_n^{\#-1} \quad (\text{A9})$$

(compare Eqs. (45) and (46)).

According to Sect. 1 the probability that a particle is scattered into a cone  $C$ , is given the formula

$$P(C, \sigma) = \lim_{t \rightarrow \infty} \lambda_{\sigma}(t)$$

with

$$\lambda_{\sigma}(t) = \int_C d^3x \langle \sigma_{\text{sc}}(t), \sigma_{\text{sc}}(t) \rangle(x).$$

The scattering wave  $\sigma_{\text{sc}}(t)$  is on our case a section defined by the formula

$$\sigma_{\text{sc}}(t) = \exp\{-iHt\Omega_{-}\sigma\} - \exp\{-iH_0t\sigma\}$$

for arbitrary  $\sigma \in H(n)$  (compare Sect. 2 and 3). We prove now

holds for arbitrary positive  $t$ . Now we have the inequalities:

$$|\lambda_{\sigma}(t)^{1/2} - \tilde{\lambda}_{\sigma}(t)^{1/2}| \leq \left| \int_C d^3x \langle \sigma_{\text{sc}}(t) - \tilde{\sigma}(t), \sigma_{\text{sc}}(t) - \tilde{\sigma}(t) \rangle(x) \right|^{1/2} \leq |\sigma_{\text{sc}}(t) - \tilde{\sigma}(t)|,$$

whence

$$\lim_{t \rightarrow \infty} |\lambda_{\sigma}(t)^{1/2} - \lambda(\sigma)^{1/2}| = \lim_{t \rightarrow \infty} |\lambda_{\sigma}(t)^{1/2} - \tilde{\lambda}_{\sigma}(t)^{1/2}| \leq \lim_{t \rightarrow \infty} |\sigma_{\text{sc}}(t) - \tilde{\sigma}(t)|.$$

From the explicit definitions of  $\sigma_{\text{sc}}(t)$  and  $\tilde{\sigma}(t)$  and the fact that  $d_n^{\#}$  is an isometry we find that

$$|\sigma_{\text{sc}}(t) - \tilde{\sigma}(t)| = |e^{-i\tilde{H}t} \tilde{\Omega}_{-} d_n^{\#-1} \sigma - e^{-i\tilde{H}_0t} d_n^{\#-1} \sigma - V_{+}^0(t) d_n^{\#-1} (S-1) \sigma|$$

which vanishes by virtue of Lemma 2 and 3 when  $t \rightarrow \infty$ . This yields immediately the Equation (A10).

Comparing  $P(C, \sigma)$  with the corresponding expression of ordinary potential scattering [7], we see that  $F\sigma$  replaces the Fourier transformation, which cannot be applied in our case. We use, however, this analogy in order to identify  $F\sigma$  with the “momentum space”-wave-function of quantum mechanics (note, however, that  $F\sigma$  is a section). Hence

lemma 4.

$$P(C, \sigma) = \int_C d^3x \langle (S-1)F\sigma, (S-1)F\sigma \rangle(x), \quad (\text{A10})$$

where  $F$  is the unitary operator

$$F = d_n^{\#} U_0 T_0 d_n^{\#-1},$$

(compare Equation (48)).

*Proof.* Using the invariance of the cone  $C$  under dilatations, it is easily established that for arbitrary  $\sigma \in H$  and positive real  $\alpha$  the quantities

$$\int_C d^3x \langle d_n^{\#} U_i(\alpha) d_n^{\#-1} \sigma, d_n^{\#} U_i(\alpha) d_n^{\#-1} \sigma \rangle(x)$$

are independent of  $\alpha$  and equal to

$$\int_C d^3x \langle \sigma, \sigma \rangle(x).$$

Using this fact and the commutativity of  $F$  and  $S$  one finds by inspection of Eq. (A2) that the formula

$$\begin{aligned} \lambda(\sigma) &= \int_C d^3x \langle (S-1)F\sigma, (S-1)F\sigma \rangle(x) \\ &= \tilde{\lambda}_{\sigma}(t) \end{aligned}$$

with

$$\tilde{\lambda}_{\sigma}(t) = \int_C d^3x \langle \tilde{\sigma}(t), \tilde{\sigma}(t) \rangle,$$

and

$$\tilde{\sigma}(t) = d_k^{\#} V_0^+(t) d_k^{\#-1} (S-1) \sigma,$$

$\langle F\sigma, F\sigma \rangle(p)$  becomes the probability that our particle has momentum  $p$ .

Assume now that  $\langle F\sigma, F\sigma \rangle(p)$  has support in a cone  $C'$ . If  $C$  and  $C'$  are sufficiently narrow there is a section  $\tau: C \cup C' \rightarrow P$  (compare Sect. 3) and, by formula (23), we have the identity

$$\langle F\sigma, F\sigma \rangle(p) = \bar{\varphi}(p) \varphi(p)$$

with

$$\varphi(p) = (d_n^{\#-1} F\sigma)(\tau(p)).$$



$\varphi(p)$  is an ordinary function, which we may identify with the momentum space wave function in the normal sense. (Note, however, the dependence on  $\tau$ .)

A rather lengthy computation which relies on the orthogonality properties of the Wigner coefficients  $D_{m'm'}^j$ , shows that

$$\int_C d^3p \langle (S-1)F\sigma, (S-1)F\sigma \rangle(p) = \int_C d^3p \int_{S^2} d^2p' \frac{P}{2\pi i} f(p_0, p_0') \varphi(|p|p_0')|^2, \quad (\text{A11})$$

where the second integration is over the 2-sphere,

$$p_0 = p/|p|,$$

$$f(p_0, p_0') = \frac{i}{2|p|} \sum_{k=0}^{\infty} T(k) (|m'| + k + 1) D_{m', m'}^{|m'|+k}(\tau(p_0') \cdot \tau(p_0)), \quad (\text{A12})$$

and

$$T(k) = -1 + \exp\{-i\pi(|m'| + k + \frac{1}{2})^2 - m'^2)^{1/2} - |m'| - k - \frac{1}{2}\}.$$

If we compare (A11) again with the “scattering into cones” formula of ordinary potential scattering [7] we see that  $f(p_0, p_0')$  has to be identified with the scattering amplitude. Hence the differential cross-section for particles of momentum  $|p|p_0'$  is given by the formula

$$d\sigma/d\Omega = |f(p_0, p_0')|^2. \quad (\text{A13})$$

In contrast to  $f(p_0, p_0')$ , the differential cross-section is found to be independent of  $\tau$ . Moreover, it is rotationally invariant. Hence we may assume without loss of generality that

$$p_0' = e_3, \quad p_0 = \cos\theta e_3 + \sin\theta e_1,$$

where  $\theta$  is the scattering angle. Since  $d\sigma/d\Omega$  is independent of  $\tau$ , any choice of  $\tau(p_0)$  and  $\tau(p_0')$  which projects down to  $p_0$  and  $p_0'$ , respectively, will yield the same expression for  $d\sigma/d\Omega$ . We choose

$$\tau(p_0') = e_0, \quad \tau(p_0) = \exp(\frac{1}{2}\theta e_2),$$

(compare Section 3). Using the definition of  $D_{m', m'}^j$ , we find that for this choice [15]

$$D_{m', m'}^{|m'|+k}(\tau(p_0') \tau(p_0)) = (\cos \frac{1}{2}\theta)^{|n|} P_k^{(0, |n|)}(\cos\theta) \quad (\text{A14})$$

(recall that  $m' = -n/2$ ).  $P_k^{(0, |n|)}$  is a Jacobi polynomial and the functions

$$f_k^{|n|}(\theta) = \sin\theta^{1/2} (\cos \frac{1}{2}\theta)^{|n|} P_k^{(0, |n|)}(\cos\theta)$$

satisfy the differential equation [19]

$$\frac{d^2}{d\theta^2} f_k^{|n|} + \left[ \frac{1}{16 \sin^2 \frac{1}{2}\theta} + \frac{1 - 4n^2}{16 \cos^2 \frac{1}{2}\theta} + \left( k + \frac{|n| + 1}{2} \right)^2 \right] f_k^{|n|} = 0. \quad (\text{A15})$$

Equation (49) is then found by combining Eq. (A13) with (A14) and (A15).

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